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A note on the multiple elliptic Dedekind sum

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1 Introduction

An elliptic analogue of the *multiple Dedekind sum* was introduced in [E], and reported in the talk of the author at this(RIMS, 1994) conference. In this note we generalize the above results in a form suggested by professors Akiyama, Tanigawa, and Wakabayashi after the conference. The author would like to appreciate their valuable suggestion.

2 Notations

Let τ be an element of the complex upper half plane. Set $\omega_1 = \pi i, \omega_2 = -\pi i(1+\tau)$, and $\omega_3 = \pi i\tau$. We denote by $\wp(\tau, z)$ the Wierstrass $\wp$-function associated to the lattice $L = L_\tau = \mathbf{Z}2\pi i+\mathbf{Z}2\pi i\tau$. And for $i = 1, 2, 3$ we set $\Theta_i = 2\omega_i, \Omega_i = 2\omega_{i+2}$, (should be interpreted mod. 3), $L_i = \mathbf{Z}2\Omega_i + \mathbf{Z}\Theta_i$. Note that $L = \mathbf{Z}\Omega_i + \mathbf{Z}\Theta_i$ for any $i = 1, 2, 3$.

We define elliptic functions $\varphi_i(\tau, z)$ associated to the lattices L_i by the following two conditions:

$$\varphi_i(\tau, z)^2 = \wp(\tau, z) - e_i(\tau), \tag{1}$$

$$\varphi_i(\tau, z) = \frac{1}{z} + \cdots, \tag{2}$$

where $e_i(\tau) = \wp(\tau, \omega_i)$.

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As usual we denote

$$g_2(\tau) = 60 \sum_{w \in L-\{0\}} \frac{1}{w^4},$$
$$g_3(\tau) = 140 \sum_{w \in L-\{0\}} \frac{1}{w^6}.$$

Then it is easily seen

Proposition. *For $i = 1, 2, 3$,*

1.

$$\varphi_i(\tau, z + \Omega_i) = -\varphi_i(\tau, z), \quad \varphi_i(\tau, z + \Theta_i) = \varphi_i(\tau, z), \quad \varphi_i(\tau, -z) = -\varphi_i(\tau, z)$$

2. At $z = 0$

$$\varphi_i(\tau, z) = \frac{1}{z}(1 + H_2^{(i)} z^2 + H_4^{(i)} z^4 + \cdots)$$

, where $H_{2n}^{(i)}$ is a polynomial in g_2, g_3, and e_i with rational coefficients.

We define polynomials $M_{n,r}^{(i)}(t_0, \ldots, t_r)$ with coefficints in $\mathbf{Q}[e_i, g_2, g_3]$ by

$$z^{r+1} \prod_{k=0}^{r} \varphi_i(\tau, t_k z) = \sum_{n=0}^{\infty} M_{n,r}^{(i)}(t_0, \ldots, t_r) z^{2n}.$$

3 Multiple elliptic Dedekind sum and the reciprocity

Let p be a natural number and let $a_1, \ldots, a_r$ integers coprime to p such that $p + a_1 + \cdots + a_r$ is even. We define the *multiple elliptic Dedekind sum* by

$$D_r^{(i)}(p : a_1, \ldots, a_r) = \sum_{\substack{m,n=0 \\ (m,n) \neq (0,0)}}^{p-1} (-1)^m \prod_{k=1}^{r} \varphi_i(\tau, \frac{a_k}{p}(m\Omega_i + n\Theta_i)).$$

Then we have

Theorem. (Reciprocity law) *Let $a_0, a_1, \ldots, a_r$ be pairwise coprime natural numbers such that $a_0 + a_1 + \cdots + a_r$ is even. Then*

$$\sum_{k=0}^{r} \frac{1}{a_k} D_r^{(i)}(a_0, \ldots, a_{k-1}, a_{k+1}, \ldots, a_r) = -M_{r,r}^{(i)}(a_0, \ldots, a_r).$$

The proof is similar to **Theorem 1** of [E].

4 Remarks

In [E] and the talk we treated only $D_\tau^{(1)}$ without assigning index "1". As was explained there, $D_\tau^{(1)}$ has a relation to the multiple Dedekind sum in the sense of [Z] and $M_{r,\tau}^{(1)}$ has a significance in topology [HBJ]. We do not know the corresponding results for indices 2,3.

References

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